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Derivative Domains of Rationality.

BY S. EPSTEEN.

INTRODUCTION.

In this paper I have attempted to outline a theory of derivative domains of rationality and to give a few applications to linear homogeneous differential equations.

§ 1, 3°, is based on a paper of E. H. Moore; the points of difference being (a) the function theoretic domains may be founded on any of the five precisely defined number domains instead of only one, the domain of rationals; (b) the assumptions regarding the functions are such as to lead to Fuchs' existence theorem for linear homogeneous differential equations; and (c) it is shown how a domain may be enlarged by the adjunction of *operations* as well as *functions.* § 2 is based on the work of Frobenius, Koenigsberger and Loewy. The irreducibility and reducibility definitions of Frobenius and Koenigsberger overlap when the latter are applied to linear homogeneous equations, and this leads to confusion. The discriminating notation of § 2, 1° and 2°, removes the difficulty; in § 2, 3°, this is extended to Loewy's relative irreducibility and relative reducibility, and concrete illustrations are given.

In all cases when theorems due to others are used or restated from the new standpoint, due credit is given in the foot-notes to the respective authors. The remaining portions of the paper (excepting a few points of common knowledge, such as § 1, 1° and 5°) I believe to be new.

It may be well to call the attention of the reader to the analogies with Algebra. Indeed, the Normal Domains (§ 3) and the Index of an Equation (§ 4) were originally suggested in this manner.

§ 1. CONSTRUCTION OF THE DOMAINS.

1°. *The Numbers of the Domains.* A set of numbers is called a *domain* (*realm, field*) *of rationality,* or briefly, a *domain,* when all the sums, differences, products and quotients (division by zero excepted) are numbers belonging to the set. All rational numbers (the positive and negative integers and fractions) constitute a domain which we shall designate by the black-faced letter **r**. The integers by themselves do not form a domain, because the quotient of two integers may be fractional. The positive numbers by themselves do not form a domain, because the difference of two positives may be negative.

The numbers of the domain **r**, together with $i = \sqrt{-1}$ ($a_1 + ib_1$, where a_1 and b_1 belong to **r**), form a domain **c**.

The totality of real numbers forms a domain **R**, the *domain of reals*. The numbers of **R**, together with $i = \sqrt{-1}$ ($A_1 + iB_1$, where A_1 and B_1 belong to **R**), form a domain **C**, the *complex domain*.

Since the sum, difference, product and quotient of two algebraic numbers* is an algebraic number, the totality of algebraic numbers forms a domain **A**.

The above five domains are given as illustrations; they are not exhaustive, since other domains may be constructed.

2°. *The Functions of the Domains.* Throughout this paper it will be assumed that all the functions of the complex variable $x = x_1 + ix_2$, which we consider, are analytic and meromorphic in a certain common finite region of existence X of the x-plane, the poles being isolated points.† These two assumptions are made in order that we may be able to assert with Fuchs ‡ that every linear homogeneous differential equation

$$K(y) \equiv \frac{d^n y}{dx^n} + k_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + k_n(x)\, y = 0,$$

with coefficients in the given domain, has a fundamental system of integrals $y_1 \cdots\cdot y_n$, every one of whose elements is holomorphic and analytic in X except at isolated points, and is determined by its value and the values of its first $n-1$ derivatives at any arbitrary regular point $\bar{x}$ of X.

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* An algebraic number is the root of an algebraic equation; *i. e.*, of an equation in integral powers of a single unknown with coefficients belonging to **r**.

† The totality of these singularities does not have to be a system of isolated points; the requirement is merely that the singular points of any one function be isolated.

‡ L. Fuchs: "Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten," *Journal für reine und angewandte Mathematik*, Vol. LXVI (1866), pp. 121–160.

3°. *Function-Theoretic Domains.** Take any m functions

$$u_j = u_j(x), \qquad (j = 1, \ldots, m),$$

and consider the totality of rational functions of these, the coefficients being numbers of **C**.† This totality constitutes the *function-theoretic domain of rationality*

$$\mathbf{C}<x> \equiv \mathbf{C}[u_1 \ldots u_m].$$

We say that $\mathbf{C}<x>$ is obtained from **C** by the *adjunction* of the functions $u_1 \ldots u_m$.

Certain functions of a domain, without vanishing formally, that is, for $u_1 \ldots u_m$ as indeterminates, vanish identically as functions of x; these equations are the *identities* of the domain. For example, the expression $1 - u_1^2 - u_2^2$ does not vanish when u_1 and u_2 are regarded as indeterminates, but it becomes identically zero for $u_1 = \sin x$, $u_2 = \cos x$.

If in addition to the m functions u_j of $\mathbf{C}<x>$ certain M additional functions $U_J(x)$ are introduced and these $m + M$ functions are utilized for the construction of a new domain $\bar{\mathbf{C}}<x>$, then $\bar{\mathbf{C}}<x>$ contains $\mathbf{C}<x>$ as a *subdomain.* One sees easily that **r** is a subdomain of **c** and of **R**; and that **r**, **c** and **R** are all subdomains of **C**.

As before, it is said that $\bar{\mathbf{C}}<x>$ is obtained from $\mathbf{C}<x>$ by the *adjunction* of the M functions U_J, or, $\bar{\mathbf{C}}<x>$ is obtained from **C** by the adjunction of the $m + M$ functions u_j, U_J.

If to the domain $\mathbf{C}<x>$ the first derivatives

$$\frac{d u_j}{d x} = u_j', \qquad (j = 1, \ldots, m),$$

of the functions $u_j(x)$ are adjoined, the *first derivative domain* is obtained,

$$\mathbf{C}^{(1)}<x> \equiv \mathbf{C}^{(1)}[u_1 \ldots u_m] \equiv \mathbf{C}[u_1 \ldots u_m, u_1' \ldots u_m'];$$

and similarly, by adjunction of all derivatives of order 1, 2, ..., h, the h-th derivative domain

$$\mathbf{C}^{(h)}<x> \equiv \mathbf{C}^{(h)}[u_1 \ldots u_m],$$

and thus an infinite succession of derivative domains ($h = 1, 2, \ldots$) is obtained.

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* E. H. Moore: "Concerning Transcendentally Transcendental Functions," *Mathematische Annalen*, Vol. XLVIII.

† For some investigations the original number domain might well be **r**, **c**, **R** or **A**. It is an interesting problem, in any specific investigation, to define the smallest domain in which the results are valid.

If the m first derivatives $u'_1 \dots u'_m$ belong to the original domain $\mathbf{C}<x>$, then the derivative of every function of $\mathbf{C}<x>$ belongs to $\mathbf{C}<x>$ and the original domain is identical with the first derivative domain, and, indeed, with every derivative domain $\mathbf{C}^{(h)}<x>$, $(h = 1, 2, \dots)$. In this case the domain $\mathbf{C}<x>$ is said to be *perfect.*

One should distinguish between $\mathbf{C}<x>$ and $\mathbf{C}[x]$. The former contains functions of x, but does not necessarily contain the independent variable x itself; the latter is the domain $\mathbf{C}$ with x adjoined, *i. e.*, the domain of rational functions of x with coefficients belonging to $\mathbf{C}$. In a sense, $\mathbf{C}[x]$ may be regarded as a special case of $\mathbf{C}<x>$.

The following are examples of perfect domains:

$$\text{a) } \mathbf{R}; \quad \text{b) } \mathbf{C}; \quad \text{c) } \mathbf{C}[x]; \quad \text{d) } \mathbf{R}[x, e^x]; \quad \text{e) } \mathbf{C}[x, e^x];$$
$$\text{f) } \mathbf{R}[x, \log x]; \quad \text{g) } \mathbf{C}[x, \log x].$$

In the applications we shall consider none but perfect domains. Should it be desired to transform the independent variable of the equation $K(y) = 0$ from x to ξ by means of

$$x = x(\xi), \text{ or } \xi = \xi(x),$$

the perfect domain $\mathbf{C}<x> \equiv \mathbf{C}[u_1 \dots u_m]$ is transformed into

$$\mathbf{C}<x(\xi)> \equiv C_1<\xi> \equiv \mathbf{C}[g_1(\xi) \dots g_m(\xi)],$$

where

$$g_j(\xi) = u_j(x(\xi)) = u_j(x).$$

In order that $C_1<x>$ shall also be a perfect domain, it is necessary and sufficient that $d\xi/dx$ shall belong to $\mathbf{C}<x>$, or, what amounts to the same thing, $dx/d\xi$ belong to $\mathbf{C}<\xi>$.

A domain may be enlarged not only by the adjunction of functions, but also by the adjunction of *operations* (*processes*). Thus, having given a domain $\mathbf{C}<x>$, the adjunction of the operation of differentiation means that the original functions $u_1 \dots u_m$ must be such as to possess derivatives, and that the derivative of every function is in the domain.

4°. *Theory of Adjunction.* Consider the quadratic equation $ax^2 + bx + c = 0$ with coefficients in the domain $\mathbf{r}$. For brevity, we say the equation is in $\mathbf{r}$. Ordinarily, the equation is not solvable in this domain. The simplest irrationality that need be adjoined is the solution of $x^2 = \Delta_1$, where Δ_1 is any number in $\mathbf{r}$. In the domain $\mathbf{r}[\Delta_1^{\frac{1}{2}}]$ all quadratic equations are solvable.

Ordinarily, cubic equations with coefficients in **r** are not solvable in **r** or **r** $[\Delta_1^{\frac{1}{2}}]$. The simplest irrationality that need be adjoined is the solution of $x^3 = \Delta_2$, where Δ_2 is any number in **r** $[\Delta_1^{\frac{1}{2}}]$. In the domain **r** $[\Delta_1^{\frac{1}{2}}, \Delta_2^{\frac{1}{3}}]$ all cubic equations are solvable.

All quartic equations in **r** are solvable in **r** $[\Delta_1^{\frac{1}{2}}, \Delta_2^{\frac{1}{3}}]$, but quintic equations are not. It is known that the simplest irrationality which need be adjoined to render quintic equations solvable is the icosahedron irrationality.*

The question now arises, how much knowledge of a new irrationality is needed to make it known to the extent that it may be adjoined to (included in) the domain? The reply to this is a relative one, and can be gathered from some concrete illustrations. Let the reader think of a hypothetical person who is thoroughly familiar with the domain of rational numbers **r**, and who is unacquainted with the extraction of square roots. In order that this person may adjoin to the domain $u = \sqrt{a}$, where a is a positive number in **r**, he would have to learn (1°) some definition of u, and from this (2°) to expand u in a convergent series $a_0 + a_1/10 + a_2/10^2 + \ldots$ (*i. e.*, learn to approximate it in **r** to any assigned degree of accuracy), (3°) to perform on it the operations of **r**, such as $u + u$, $u \times u$, etc., (4°) to combine it with the numbers $\alpha, \beta, \ldots$, of **r**, as for instance, αu, $(\alpha u + \beta)/(\gamma u + \delta)$, $\alpha u^2 + \beta u + \gamma$. In time, the person in question would be so familiar with the square root process that he could adjoin it (or the square root of any definite positive number a) to the domain **r**.

Suppose this hypothetical person is to be taught the trigonometric functions until he can (if he so chooses) adjoin any of them, such as $\sin x$, to the domain. He would have to learn (1°) some definition of each of these functions, and from these (2°) to expand them in infinite series; (3°) the addition theorems; (4°) the algebraic relations between these functions, such as $\sin^2 x + \cos^2 x = 1$; (5°) the differential coefficients of these functions.† In brief, after a full course in Trigonometry, he could adjoin $\sin x$ to the domain of rationality, if he so desired.

Following out this thought, a person familiar with the Elliptic Functions may adjoin them to the domain of rationality in which he is solving a given (algebraic or differential) equation. In general, when we encounter a class of algebraic or differential equations giving rise to a new irrationality, we must familiarize ourselves with this new function sufficiently, and then equations dependent on it may be regarded as solved if the functional dependence is known.

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* F. Klein: "Vorlesungen über das Ikosaeder," Leipzig, Teubner (1884).

† A knowledge of the derivatives of the functions would be necessary on account of our assumption that all the domains under consideration are perfect.

5°. *Irrationality.* Summarizing the foregoing, the derivative domains of rationality used in this paper, and designated frequently by $\mathfrak{D}\langle x\rangle$, will consist of some number domain (usually **C**) to which any arbitrary functions, fulfilling the conditions of § 1, 2°, are adjoined. Every sum, difference, product, quotient (excepting the zero-divisor) and derivative will give an element of the domain. When equations are under consideration, the domain is enlarged by adjunction until it contains every coefficient at least.

An algebraic equation or an algebraic differential equation all of whose coefficients are elements of a given (derivative) domain $\mathfrak{D}\langle x\rangle$ is said to *belong to, lie in,* or *be an equation in* $\mathfrak{D}\langle x\rangle$.

A function (or number) $f(x)$ which lies in $\mathfrak{D}\langle x\rangle$ is *rational with respect to this domain*; a function which does not lie in $\mathfrak{D}\langle x\rangle$ is *irrational with respect to this domain.* Irrational functions are of two kinds: *algebraic* if they satisfy some algebraic equation in $\mathfrak{D}\langle x\rangle$; *transcendental* if they do not satisfy such an equation. Thus, e and π are transcendental (numbers) with respect to the (number) domains **r**, **c**. Transcendental functions are of two kinds: *algebraically transcendental** *with respect to* $\mathfrak{D}\langle x\rangle$ if they satisfy some algebraic differential equation in this domain, and *transcendentally transcendental* if they do not satisfy such an equation. Thus e^x is algebraically transcendental with respect to $\mathbf{r}[x]$, being a solution of $y' - y = 0$, while the well-known gamma function, $\Gamma(x)$, is transcendentally transcendental with respect to the same domain.†

§ 2. THE CONCEPTS OF IRREDUCIBILITY AND REDUCIBILITY IN PRESCRIBED DOMAINS.

1°. *Frobenius' Irreducibility and Reducibility.* If a linear homogeneous differential equation in a domain $\mathfrak{D}\langle x\rangle$ of order n, $K(y)=0$, does not have an integral in common with another linear homogeneous differential equation in $\mathfrak{D}\langle x\rangle$ of order $n_1 < n$, we shall call it *f-irreducible* (*f-i*) *in* $\mathfrak{D}\langle x\rangle$. An equation which is not *f-i* in $\mathfrak{D}\langle x\rangle$ is *f-reducible*‡ (*f-r*).

That the irreducibility or reducibility of an equation depends on the domain of rationality is easily illustrated. Thus, $y''+y=0$ is reducible in the domain **C**, because the integral $y_1 = e^{ix}$ satisfies the equation (in **C**) $y' - iy = 0$; however, the equation is irreducible in **R**.

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* E. H. Moore, *loc. cit.*, p. 52.

† O. Hölder: "Ueber eine Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen," *Mathematische Annalen*, Vol. XXVIII (1887), pp. 1–13.

‡ After Frobenius, *Crelle's Journal für reine und angewandte Mathematik*, Vol. LXXVI.

There are many theorems on reducible differential equations analogous to well-known theorems on reducible algebraic equations, but occasionally the analogy breaks down. For instance, the fact that every root of a (reducible) algebraic equation satisfies *some* irreducible algebraic equation of lower degree has no counterpart. An integral of a reducible differential equation does not necessarily satisfy some irreducible equation of lower order. Thus, the equation $y''' - 4y'' + 5y' - 2y = 0$ is reducible in **R**; its integrals $y_1 = e^{2x}$, $y_2 = e^x$ satisfy the irreducible equations $y' - 2y = 0$, $y' - y = 0$, but the integral $y_3 = xe^x$ does not satisfy any equation irreducible in **R** and of order lower than 3. Indeed, the equation of lowest order which it satisfies is a reducible one:

$$y'' - 2y' + y = 0.$$

A differential equation with the special property of possessing a fundamental system of integrals $y_1 \dots . y_n$ each of which satisfies an irreducible equation, has been called *completely reducible.**

2°. *Koenigsberger's Irreducibility and Reducibility.* We will call an algebraic differential equation of order n and degree d,

$$E_{nd} \equiv e_0(x y y' \dots . y^{(n-1)})(y^{(n)})^d + e_1(x y y' \dots . y^{(n-1)})(y^{(n)})^{d-1} + \dots . + e_d(x y y' \dots . y^{(n-1)}) = 0,$$

prime in $\mathfrak{D}\langle x \rangle$, if, considered as an integral function of $y y' \dots . y^{(n)}$, it is algebraically irreducible in $\mathfrak{D}\langle x \rangle$; that is, if it is not the product of integral functions of the same arguments with coefficients in $\mathfrak{D}\langle x \rangle$. The same thought may be restated in the following equivalent manner:

a) Since each integral function of $y, y', \dots, y^{(n-1)}$ contained in E_{nd} must be a common divisor of $e_0, e_1, \dots, e_d$, the coefficients $e_0, \dots, e_d$ can not have a common factor; and

b) E_{nd} considered as a polynomial in $y^{(n)}$ must be algebraically irreducible.

An algebraic differential equation $E_{nd} = 0$ in $\mathfrak{D}\langle x \rangle$, which is prime and does not have an integral in common with another algebraic differential equation of order $n_1 < n$ in $\mathfrak{D}\langle x \rangle$ is said to be *k-irreducible*† (*k-i*) *in* $\mathfrak{D}\langle x \rangle$. An equation which is not *k-i* in $\mathfrak{D}\langle x \rangle$ is *k-reducible* (*k-r*) in this domain.

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*This concept is due to A. Loewy, *Mathematische Annalen*, Vol. LXII, who uses the term *vollständig reduzibel.* This memoir contains a number of important propositions on completely reducible equations.

†After Koenigsberger: "Allgemeine Untersuchungen aus der Theorie der Differentialgleichungen," Leipzig, Teubner (1882).

Koenigsberger's definition of reducibility, applied to a linear homogeneous differential equation, becomes: a linear homogeneous differential equation of order n is *k-r* if it has an integral in common with any algebraic differential equation of order $n_1 < n$, both equations lying in $\mathfrak{D} < x >$. In particular, an equation having an algebraic integral is reducible, because the algebraic functions satisfy a differential equation of zero order (*i. e.*, an algebraic equation).

A linear homogeneous equation may be *f-i* and *k-r* at the same time. Thus, the equation $y'' + y = 0$ in the domain $\mathbf{R}$ is *f-i* but it is *k-r*, having the integral $y_1 = \sin x$ in common with $(y')^2 + y^2 - 1 = 0$.

An equation which is *f-r* is also *k-r*; an equation which is *k-i* is surely *f-i*.

It should be noticed that *f-r* and *f-i* never apply to non-linear equations,* but *k-r* and *k-i* apply to both linear and non-linear equations.

When dealing with linear homogeneous equations, we shall say, briefly, that they are irreducible or reducible, meaning thereby *f-i* and *f-r*. When *k-i* and *k-r* are intended in connection with such equations, it will be specifically stated.

3°. *Relative Irreducibility and Reducibility.*† An algebraic differential equation $E_{na}(y) = 0$ in $\mathfrak{D} < x >$ is said to be *k-irreducible* (*k-i*) *relative to any one of its integrals* y_1 if it is prime in the domain and if y_1 does not satisfy an algebraic equation in $\mathfrak{D} < x >$ of order $n_1 < n$. A linear homogeneous equation is said to be *f-irreducible* (*f-i*) *relative to any one of its integrals* y_1 if y_1 does not satisfy a linear homogeneous equation of lower order and in the same domain. In the contrary cases, we say, respectively, that the equations are *k-r* or *f-r* relative to the integral y_1.

In the domain $\mathbf{R}$ the equation

$$y'' - 2y' + y = 0$$

is *f-r* relative to $y_1 = e^x$ but it is *f-i* relative to $y_2 = xe^x$. The equation

$$(1 + x^2)\, y'' + 2xy' - 2y = 0$$

in the domain $\mathbf{R}[x]$ is *f-r* relative to $y_1 = x$, since this integral satisfies $xy' - y = 0$; it is *f-i* relative to $y_2 = 1 + x \operatorname{arc\,tan} x$; however, it is *k-r* relative to y_2, since this function satisfies the non-homogeneous equation

$$x(1 + x^2)\, y' - (1 + x^2)\, y + 1 = 0.$$

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* One could, of course, call a non-linear equation *f-r* if it has an integral in common with a linear homogeneous equation of lower order; otherwise *f-i*. But I know of no application of this extension of the concept.

† A. Loewy: "Algebraische und gruppentheoretische Untersuchungen aus dem Gebiete der Differentialgleichungen," *Monatshefte für Mathematik und Physik*, Vol. VIII (1897), pp. 227–234.

A non-linear equation of the first order, which has an algebraic and a transcendental integral, is reducible relative to the former, which satisfies an algebraic equation (a differential equation of order zero), and is irreducible relative to the transcendental integral.

If $E_{nd}(y) = 0$ in $\mathfrak{D}<x>$ is *k-i* relative to an integral y_1, we say that y_1 is a *superior integral*[*] of the equation in this domain. An integral which satisfies

$$\frac{\partial E}{\partial y^{(n)}} = 0$$

is called *singular*. A non-singular integral which is not superior is called an *inferior* integral.

4°. *Koenigsberger's Theorem.*[†] The following theorem, due to Koenigsberger, is restated without proof in the terminology of this section. An application will be made in § 3.

THEOREM I. *If a superior integral of an equation in* $\mathfrak{D}<x>$, $E_{nd}(y) = 0$, *satisfies any algebraic differential equation also in* $\mathfrak{D}<x>$, $F_{\nu d_1}(y) = 0$, *where* $\nu > n$, *then all the non-singular integrals of* $E = 0$ *satisfy* $F = 0$.

§ 3. NORMAL DOMAINS.

If the linear homogeneous differential equation

$$K(y) \equiv y^{(n)} + k_1(x) \, . \, y^{(n-1)} + \dots + k_n(x) \, . \, y = 0$$

is irreducible in $\mathfrak{D}<x>$, it follows that none of its integrals belong to $\mathfrak{D}<x>$. For, if an integral y_1 belongs to $\mathfrak{D}<x>$, then $\eta = y_1'/y_1$ also belongs to this domain, and y_1 satisfies an equation in $\mathfrak{D}<x>$,

$$H(y) \equiv \frac{dy}{dx} - \eta y = 0,$$

of the first order. This is impossible if $K(y) = 0$ is irreducible.

In case an integral y_1 does belong to $\mathfrak{D}<x>$, and, therefore, satisfies the equation $H(y) = 0$, the equation $K = 0$ is reducible, and can be factored (symbolically in the sense of Boole) into

$$K(y) \equiv G\{H(y)\}$$

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[*] Superior integrals correspond to primitive roots in Algebra. I prefer this term in connection with differential equations, since "primitive" is used in § 3 in a different sense.

[†] L. Koenigsberger, *loc. cit.* Also A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 231.

(written for brevity $G\,H(y)$), where

$$G \equiv \frac{d^{n-1}}{d\,x^{n-1}} + g_1 \frac{d^{n-2}}{d\,x^{n-2}} + \cdots\cdots + g_{n-1},$$

the $g_1 \cdots\cdots g_{n-1}$ belonging to $\mathfrak{D}<x>$. This is easily proved by calculation from the identity $K(y) = G\,H(y)$,

$$k_1 = g_1 - \eta, \qquad k_2 = g_2 - g_1 \eta - (n-1)\,\eta',$$

$$k_3 = g_3 - g_2 \eta - (n-2)\,g_1 \eta' - \frac{(n-1)(n-2)}{2!}\,\eta'',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

whence

$$g_1 = k_1 + \eta, \qquad g_2 = k_2 + k_1 \eta + \eta^2 - (n-1)\,\eta',$$

$$g_3 = k_3 + k_2 \eta + k_1(\eta^2 - \eta') + \eta^3 + n\,\eta\,\eta' + \frac{(n-1)(n-2)}{2!}\,\eta^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and it is thus seen that $g_1, g_2, \cdots\cdots, g_{n-1}$, being rational in terms of $k_1 \cdots\cdots k_n$, $\eta, \eta', \eta'', \cdots\cdots$, must lie in $\mathfrak{D}<x>$.

In general, if $K(y)$ is reducible, it can be factored symbolically into at least two factors $K = L\,(M)$,

$$L\,(y) \equiv \frac{d^{n_1} y}{d\,x^{n_1}} + l_1 \frac{d^{n_1-1} y}{d\,x^{n_1-1}} + \cdots\cdots + l_{n_1} y,$$

$$M(y) \equiv \frac{d^{n-n_1} y}{d\,x^{n-n_1}} + m_1 \frac{d^{n-n_1-1} y}{d\,x^{n-n_1-1}} + \cdots\cdots + m_{n-n_1} y,$$

where $l_1 \cdots\cdots l_{n_1}$, $m_1 \cdots\cdots m_{n-n_1}$ are functions in $\mathfrak{D}<x>$ and $n_1 < n$.*

The domain $\mathfrak{D}<x, y_1>$ will be said to be of the n-th order when it is obtained by adjoining an integral y_1 of the equation of n-th order $K(y) = 0$, irreducible in $\mathfrak{D}<x>$.

If n particular integrals $y_1 \cdots\cdots y_n$ are adjoined to $\mathfrak{D}<x>$, we obtain n domains

$$\mathfrak{D}<x, y_1> \cdots\cdots \mathfrak{D}<x, y_n>,$$

which are said to be *conjugate to each other.* The domains may be all different; some, or all of them, may be identical. Thus, the equation

$$(1 - x^2)\frac{d^2 y}{d\,x^2} - x\frac{d\,y}{d\,x} - y = 0,$$

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* Schlesinger, "Handbuch der Theorie der linearen Differentialgleichungen," Vol. I, pp. 42–46, 81–87.

of which $y_1 = e^{\sin^{-1}x}$, $y_2 = e^{\cos^{-1}x}$ are integrals, is irreducible (*f-i*) in the domain $\mathfrak{r}[x]$. The conjugate domains

$$\mathfrak{r}[x, e^{\sin^{-1}x}] \text{ and } \mathfrak{r}[x, e^{\cos^{-1}x}]$$

are different. However, starting with $\mathbf{R}[x]$, the conjugate domains

$$\mathbf{R}[x, e^{\sin^{-1}x}] \text{ and } \mathbf{R}[x, e^{\cos^{-1}x}]$$

are identical, since $1/y_1 = y_2 e^{-\frac{\pi}{2}}$. The equation

$$\frac{d^2 y}{d x^2} + y = 0,$$

of which $y_1 = \sin x$ and $y_2 = \cos x$ are integrals, is irreducible in $\mathbf{R}$, but the domains

$$\mathbf{R}[y_1] \text{ and } \mathbf{R}[y_2]$$

are identical, since $dy_1/dx = y_2$ and $dy_2/dx = -y_1$.

A domain $\mathfrak{D}<y_2>$ may be equivalent to $\mathfrak{D}<y_1>$ while $\mathfrak{D}<y_1>$ may be inequivalent to $\mathfrak{D}<y_2>$. For instance, if $y_2 = y_1^2$, then $\mathfrak{D}<y_2>$ is equivalent to $\mathfrak{D}<y_1>$, but the inverse is not true. Thus, equivalence of domains is not reciprocal; if $\mathfrak{D}_1 \equiv \mathfrak{D}_2$, it does not follow that $\mathfrak{D}_2 \equiv \mathfrak{D}_1$.

If all the conjugates of a given domain $\mathfrak{D}<y_1>$ are equivalent to it, then $\mathfrak{D}<y_1>$ is said to be a *normal domain.*

If $\mathfrak{D}<x, y_1>$ is a normal domain, it follows that y_2 is in this domain, and therefore that y_2 is a rational differential function of y_1,

$$y_2 = \phi_2(y_1, y_1', \ldots).$$

Similarly,

$$y_3 = \phi_3(y_1, y_1', \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$y_n = \phi_n(y_1, y_1', \ldots),$$

the ϕ's denoting rational functions of the indicated arguments.

THEOREM II. *Let $K(y) = 0$ be an equation in $\mathfrak{D}<x>$ of order n, such that the adjunction of a certain particular integral y_1 produces a normal domain $\mathfrak{D}<x, y_1>$; then the general integral of $K(y) = 0$ may be expressed as a rational differential function of a single particular integral (and n parameters).*

The general integral of $K(y)=0$ being

$$y=\alpha_1 y_1+\alpha_2 y_2+\ldots\ldots+\alpha_n y_n,$$

the α's denoting parameters, we have

$$\begin{aligned} y &= \alpha_1 y_1+\alpha_2\phi(y_1, y_1'\ldots\ldots)+\ldots\ldots+\alpha_n\phi_n(y_1, y_1'\ldots\ldots) \\ &= \psi(y_1, y_1'\ldots\ldots\,|\,\alpha_1\ldots\ldots\alpha_n). \end{aligned}$$

THEOREM III. *Let*

$$A(y)\equiv\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+a_n y=0$$

be an equation with constant coefficients, and let the domain of rationality be **C**. *If the auxiliary equation of* $A(y)=0$,

$$u^n+a_1u^{n-1}+\ldots\ldots+a_n=0,$$

has distinct roots, then the domain **C** $[V]$, *where*

$$V=y_1+y_2+\ldots\ldots+y_n,$$

is normal.

We designate the roots of the auxiliary equation by

$$u_1,\ u_2,\ \ldots\ldots,\ u_n;$$

then

$$y_i=e^{u_i x},\qquad (i=1,\ \ldots\ldots,\ n),$$

and

$$V=e^{u_1x}+e^{u_2x}+\ldots\ldots+e^{u_nx}.$$

By differentiation

$$\begin{aligned} V' &= u_1e^{u_1x}+\ldots\ldots+u_ne^{u_nx}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ V^{(n-1)} &= u_1^{n-1}e^{u_1x}+\ldots\ldots+u_n^{n-1}e^{u_nx}. \end{aligned}$$

The determinant

$$\begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ u_1 & u_2 & \ldots\ldots & u_n \\ \ldots & \ldots & \ldots\ldots & \ldots \\ u_1^{n-1} & u_2^{n-1} & \ldots\ldots & u_n^{n-1} \end{vmatrix}$$

can not vanish, the u_i being unequal. We can therefore find the y_i in terms of V,

$$y_i=b_0V+b_1V'+\ldots\ldots+b_{n-1}V^{(n-1)};$$

the $b_1\ldots\ldots b_{n-1}$ denoting numbers in **C**. Since V is itself an integral of $A(y)=0$, the domain **C** (V) must be a normal one.

THEOREM IV. *If two roots of the auxiliary equation of* $A(y)=0$ *are equal,* $u_{n-1}=u_n$, *then* $\mathrm{C}\,[x, V_{n-1}]$ *is a normal domain,*

$$V_{n-1}=y_1+y_2+\cdots\cdots+y_{n-1}.$$

The proof of this theorem and the generalization to the case where m roots of the auxiliary equation are equal is similar to the proof of Theorem III, and will present no special difficulty to the reader.

An equation of the n-th order (linear or non-linear) whose general integral is a rational differential function in $\mathfrak{D}\langle x\rangle$ (with n parameters) of a single particular superior integral, is called a *normal equation.*

The following is an immediate consequence of this definition:

THEOREM V. *By adjoining a certain particular integral* y_1 *of a normal equation to* $\mathfrak{D}\langle x\rangle$, *the enlarged domain* $\mathfrak{D}\langle x, y_1\rangle$ *is normal.*

THEOREM VI. *Let* $E(y)=0$ *be a normal equation,* y_1 *a superior integral and* $y=\phi(y_1, y_1', \ldots\ldots)$ *the general integral. If* y_2 *is any other integral of* $E(y)=0$, *then* $\phi(y_2, y_2', \ldots\ldots)$ *is also an integral of* $E(y)=0$.

Consider the equation

$$E(\phi(z, z', \ldots\ldots))=0.$$

The equation $E(y)=0$ has the superior integral y_1 in common with it; therefore (Theorem I) every integral, as y_2, of $E(y)=0$ satisfies it also. From this we see that

$$E(\phi(y_2, y_2', \ldots\ldots))=0,$$

and thus

$$\phi(y_2, y_2', \ldots\ldots)$$

is also an integral of $E(y)=0$.*

A rational integral differential function in $\mathfrak{D}\langle x\rangle$ of the fundamental system $y_1\ldots\ldots y_n$ of $K(y)=0$,

$$V=\pi(y_1\ldots\ldots y_n, y_1'\ldots\ldots y_n', \ldots\ldots),$$

written for brevity

$$V=\pi(y),$$

is said to be *primitive*† in this domain when the y's are also rational integral functions in $\mathfrak{D}\langle x\rangle$ of $V, V', \ldots\ldots$,

$$y_i=X_i(V, V', V'', \ldots\ldots), \qquad (i=1, \ldots\ldots, n).$$

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* E. Vessiot, *Annals de l'École Normale Supérieure*, Vol. IX (1892), p. 227. See also A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, § 4.

† A. Loewy, "Die Rationalitätsgruppe einer linearen homogenen Differentialgleichung," *Mathematische Annalen*, Vol. LXV (1908), p. 136.

If we differentiate $V = \pi(y)$ for x n^2 times, there arise

$$\begin{aligned} V &= \pi(y_1 \cdots y_n,\ y'_1 \cdots y'_n,\ \cdots\cdot), \\ \frac{dV}{dx} &= \pi_1(y_1 \cdots y_n,\ y'_1 \cdots y'_n,\ \cdots\cdot), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \frac{d^{n^2}V}{dx^{n^2}} &= \pi_{n^2}(y_1 \cdots y_n,\ y'_1 \cdots y'_n,\ \cdots\cdot). \end{aligned}$$

By means of the differential equation $K(y) = 0$ derivatives of $y_1 \cdots y_n$ of order higher than $n - 1$ can be eliminated, giving us then $n^2 + 1$ equations in the n^2 functions $y_1 \cdots y_n,\ y'_1 \cdots y'_n,\ \cdots,\ y_1^{(n-1)} \cdots y_n^{(n-1)}$. Suppose that these functions can be eliminated from the first $s + 1$ of the equations ($s \leq n^2$), giving rise to the resolvent

$$M\left(V, \frac{dV}{dx}, \cdots, \frac{d^s V}{dx^s}\right) = 0.$$

The equation $M = 0$ is not necessarily k-irreducible relative to V. Among all the possible equations $M = 0$ which V satisfies, we shall consider one which is irreducible in $\mathfrak{D}<x>$ relative to V; that is, which has V for a superior integral. Let this equation be

$$N\left(V, \frac{dV}{dx}, \cdots, \frac{d^r V}{dx^r}\right) = 0.$$

The following theorem is a statement in the terminology of this section of a result due to Loewy.*

THEOREM VII. *The resolvent $N = 0$ is a normal equation.*

§ 4. THE INDEX OF A LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION.

Denote the general linear homogeneous group in n variables and n^2 parameters by

$$L:\ z_{i_1} = \sum_{i_2} a_{i_1 i_2} y_{i_2},\quad |a_{i_1 i_2}| \neq 0, \qquad (i = 1, \cdots, n).$$

Let $y_1 \cdots y_n$ be a fundamental system of the equation $K(y) = 0$ in $\mathfrak{D}<x>$, then a rational differential function of these integrals with coefficients in $\mathfrak{D}<x>$,

$$t(y) \equiv t(y_1 \cdots y_n,\ y'_1 \cdots y'_n,\ \cdots\cdot),$$

is called a *natural transcendental of the domain.*†

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* A. Loewy, "Die Rationalitätsgruppe einer linearen homogenen Differentialgleichung," *Mathematische Annalen*, Vol. LXV (1908), p. 160.

† A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 254.

By the rationality group G, of the equation $K=0$, is meant the totality of substitutions (taken from L) possessing the following double property:

a) Every natural transcendental invariant under the substitutions of G has its numerical value in $\mathfrak{D}\langle x\rangle$; and b) every natural transcendental whose numerical value lies in $\mathfrak{D}\langle x\rangle$ is numerically invariant under each substitution of G.

If r is the order of the normal resolvent (and therefore the number of parameters in the rationality group G of the equation $K=0$ *), the integer

$$\delta = n^2 - r$$

is called the *index of the equation* $K(y)=0$.

The lower the order of $N=0$, the greater is the index of $K=0$. When $\delta = n^2$, the integrals $y_1 \ldots . y_n$ are all in $\mathfrak{D}\langle x\rangle$, and the equation $K=0$ is solved (requiring at most the adjunction of an algebraic irrationality). For, when $\delta = n^2$, we have $r=0$, and $N=0$ is an algebraic equation (differential equation of order zero). The solution V of this equation is at most an algebraic irrationality, and the equations

$$y_i = X_i(V, V', \ldots .), \qquad (i=1, \ldots ., n),$$

give us a fundamental system of $K=0$.

When the adjunction of a new irrationality to $\mathfrak{D}\langle x\rangle$ makes $N=0$ k-reducible, a new normal resolvent, of lower order, is obtained, and the index is thus increased. From this point of view, the problem of solving a differential equation consists in adjoining irrationalities to $\mathfrak{D}\langle x\rangle$ until the index becomes a maximum, $\delta = n^2$.

A differential equation $K(y)=0$ is said to be *special in* $\mathfrak{D}\langle x\rangle$ if the numerical value $\alpha(x)$ of a natural transcendental $t(y)$ lies in $\mathfrak{D}\langle x\rangle$ for certain fundamental systems, but not for every one of the ∞^{n^2} fundamental systems defined by $y_1 \ldots . y_n$ and L.

THEOREM VIII. *If* $K(y)=0$ *is a special equation, its rationality group* G *is the largest subgroup of* L *which leaves* $t(y)=\alpha(x)$ *numerically invariant.*

Suppose that A and B are two substitutions leaving t numerically invariant,

$$t_A = t, \quad t_B = t.$$

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* L. Schlesinger, "Bericht über die Entwicklung der Theorie der linearen Differentialgleichungen seit 1865," *Jahresbericht der deutschen Mathematischen Vereinigung*, Vol. XVIII (1909), p. 151.

Applying B to $t_A - t = 0$, we have $t_{AB} - t_B = 0$. From this, $t_{AB} = t_B = t$, and therefore the totality of substitutions leaving t (formally and) numerically invariant form a group G. From the foregoing definition of the rationality group, any substitution of L which does not leave $t(y)$ numerically invariant can not belong to G. On the other hand, every substitution of L which leaves $t(y)$ numerically invariant does belong to G. For, every natural transcendental $f(y)$ whose value is in the domain must be of the form

$$f(y_1 \dots y_n,\ y'_1 \dots y'_n,\ \dots) = \beta(x, t, t', \dots),$$

where the function β lies in $\mathfrak{D}\langle x \rangle$; and every substitution of L leaving t numerically invariant leaves f numerically invariant and belongs to G.

This proof of the theorem involves the assumption that there exists only one natural transcendental $t(y)$ whose numerical value $\alpha(x)$ lies in the domain for some, but not for all the ∞^{n^2} fundamental systems defined by L. If there should exist a such independent natural transcendentals, we have the

COROLLARY: *If $t_1(y) = \alpha_1(x) \dots t_a(y) = \alpha_a(x)$ are natural transcendentals whose numerical values lie in $\mathfrak{D}\langle x \rangle$, then the rationality group G of $K(y) = 0$ is that subgroup of L which leaves all these natural transcendentals numerically invariant. That is to say, it is the greatest common subgroup of the groups to which $t_1 \dots t_a$ belong.*

The following theorem will be needed in the application of § 5. We refer the reader to the proof in Loewy's paper, "Algebraische und Gruppentheoretische Untersuchungen aus dem Gebiete der Differentialgleichungen," *Monatshefte für Mathematik und Physik*, Vol. VIII (1897), p. 254.*

THEOREM IX. *If the rationality group of $K = 0$ is G, the adjunction of the numerical value $\alpha(x)$ of a natural transcendental $S(y)$ lowers it to the largest subgroup of G which leaves S numerically invariant.*

We now prove†

THEOREM X. *For a special equation $K(y) = 0$, the index is greater than zero,*

$$\delta > 0.$$

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* See also E. Vessiot, *Annales de l'École Normale Supérieure*, Vol. IX (1892), p. 235.

† Theorems X and XI are closely related to those in Picard's Traité d'Analyse, Vol. III, 2d edition (1908), p. 571. There are two elements of difference. First, in this paper they are stated from the new standpoint of the index of an equation; and, secondly, the proof is based on the general primitive functions V of Loewy instead of on the special one (known as the Picard resolvent) $u_1 y_1 + \dots + u_n y_n$.

As before, let

$$V = \pi(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \ldots .)$$

be a primitive function, which, solved for $y_1 \cdots y_n$, gives

$$y_i = X_i(V, V', \ldots .), \qquad (i = 1, \ldots ., n).$$

Since $\delta = n^2 - r$, it will suffice to show that for a special equation the order r of the normal resolvent $N(V) = 0$ is less than n^2,

$$r < n^2.$$

The equation $K(y) = 0$ being special, there exists a natural transcendental $t(y)$ whose numerical value $\alpha(x)$ lies in $\mathfrak{D} < x >$,

$$t(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \ldots .) = \alpha(x).$$

Substituting the values of the y_i,

$$t[X_1(V, V', \ldots .) \ldots . X_n(V, V', \ldots .),\ \ldots .] - \alpha(x) = 0$$

may be regarded as a differential equation for V. The order s of this equation must be less than n^2; for, to each one of its integrals corresponds a fundamental system $y_1 \ldots . y_n$ of K which satisfies $t(y) = \alpha(x)$; and if the order of $t[X_1 \ldots . X_n, \ldots .] - \alpha(x) = 0$ is n^2, then $t(y)$ would equal $\alpha(x)$ for every one of the ∞^{n^2} fundamental systems of $K = 0$; but this is contrary to the hypothesis that $K = 0$ is a special equation. $N(V) = 0$ has the superior integral V in common with $t_1[X(V, V', \ldots .) \ldots .] - \alpha(x) = 0$; therefore (Theorem I) every integral of $N = 0$ satisfies this equation, and

$$r \leqq s < n^2.$$

THEOREM XI. *If the index of $K(y) = 0$ is greater than zero,*

$$\delta > 0,$$

the equation is a special one.

From $\delta = n^2 - r > 0$, *we have, for the order of the resolvent $N = 0$,*

$$r < n^2.$$

Since $N(V) = 0$ is of order $r < n^2$, the corresponding fundamental systems $y_1 \cdots y_n$,

$$y_i = X_i(V, V', \ldots .), \qquad (i = 1, \ldots ., n),$$

are ∞^r in number (and not ∞^{n^2}). If $r < n$, we can eliminate V from the n equations $y_i = X_i(V, V', \ldots .)$ and thus obtain a relation between the y_i

satisfied by some, but not by all fundamental systems. If $n \overset{=}{<} r < 2n$, consider the $2n$ equations

$$y_i = X_i(V, V', \ldots.),$$
$$y_i' = \bar{X}_i(V, V', \ldots.).$$

Remove derivatives of V beyond order r by means of $N = 0$. If $r < 2n$, we can eliminate as before, and obtain a relation between $y_1 \ldots . y_n$, $y_1' \ldots . y_n'$ true for some, but not for all fundamental systems of $K = 0$. If $2n \overset{=}{<} r < 3n$, the same process will give us a relation between $y_1 \ldots . y_n$, $y_1' \ldots . y_n'$, $y_1'' \ldots . y_n''$. Continuing in this manner, it is shown that $K = 0$ is a special equation.

§ 5. INTEGRATION OF A DIFFERENTIAL EQUATION BY CHAINS OF RESOLVENTS.

There are two methods of integrating linear differential equations by chains of resolvents. The first one (although second in historical order) is due to Loewy[*] and may be called the *Method of Normal Resolvents.* This method has the advantage that but one particular integral of each resolvent need be computed to give the general integral; its disadvantage (as compared with the second method) is that the resolvents are not of lowest possible order. The second method (first in historical order) is due to Vessiot,[†] and may be called the *Method of Minimum Resolvents.* This method has the advantage that the resolvents to be integrated are of the lowest possible order; its disadvantage (as compared with the first method) is that more than one particular integral of each resolvent must be determined to give the general integral.

In this section these two methods are restated (in modified form) as applications of the concepts of § 4.

1°. Let G be the rationality group of $K(y) = 0$ in $\mathfrak{D}<x>$, and by adjoining an algebraic irrationality[‡], f, to $\mathfrak{D}<x>$ we lower it to the maximal continuous subgroup of G, denoted by Γ, and generated by infinitesimal transformations. Let $\Gamma^{(1)}$ be a maximal self-conjugate subgroup of $\Gamma^{(0)} \equiv \Gamma$, let $\Gamma^{(2)}$ be a maximal self-conjugate subgroup of $\Gamma^{(1)}$, and so forth. By continuing this process we finally reach a simple group $\Gamma^{(m-1)}$, and the next one in the list, $\Gamma^{(m)}$, must be the identity, I. The set of groups

$$\Gamma^{(0)},\ \Gamma^{(1)},\ \ldots.,\ \Gamma^{(m-1)},\ \Gamma^{(m)}$$

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[*] A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 264.

[†] E. Vessiot, *loc. cit.*, p. 239.

[‡] L. Schlesinger, "Handbuch der Theorie der linearen Differentialgleichungen," II, 1, p. 80.

is called a *composition series* of Γ. The number of parameters in $\Gamma^{(k)}$ is designated by r_k, and we let

$$\rho_k = r_{k-1} - r_k, \qquad (k = 1, \ldots, m),$$

where $r_0 = r$, $r_m = 0$.

Let $t_1 \equiv t_1(y) \equiv t_1(y_1 \ldots y_n, y_1' \ldots y_n', \ldots)$ be a natural transcendental belonging to $\Gamma^{(1)}$; that is, an invariant of $\Gamma^{(1)}$ which is altered in value by any substitution outside of $\Gamma^{(1)}$. The function t_1, when transformed by Γ, contains ρ_1 effective parameters,* and it is a solution of a normal resolvent

$$\Phi_1(t_1) = 0,$$

of order ρ_1. Any other solution of this equation, $\bar{t}_1$, belongs to the same group $\Gamma^{(1)}$; the same is true of the general integral T_1, and we have†

$$T_1 = A_1(t_1 t_1' \ldots \mid p_1 \ldots p_{\rho_1}),$$

A_1 denoting an algebraic function, $p_1 \ldots p_{\rho_1}$ parameters.

If we determine one integral t_1 of $\Phi_1 = 0$ and adjoin it to $\mathcal{D}<x, f>$, the rationality group of $K(y) = 0$ in $\mathcal{D}<x, f, t_1>$ falls to $\Gamma^{(1)}$ (Theorem IX). The index δ of $K(y) = 0$ is increased,

$$\delta_1 = \delta + \rho_1.$$

Let t_2 be a natural transcendental belonging to $\Gamma^{(2)}$; when transformed by $\Gamma^{(1)}$ it takes ρ_2 effective parameters and it is a solution of a normal resolvent in $\mathcal{D}<x, f, t_1>$,

$$\Phi_2(t_2) = 0,$$

of order ρ_2. In the domain $\mathcal{D}<x, f, t_1, t_2>$ the group of $K = 0$ is $\Gamma^{(2)}$ and the index

$$\delta_2 = \delta_1 + \rho_2 = \delta + \rho_1 + \rho_2.$$

Similarly, in the domain $\mathcal{D}<x, f, t_1, t_2, t_3>$ the rationality group of $K = 0$ is $\Gamma^{(3)}$ and the index

$$\delta_3 = \delta + \rho_1 + \rho_2 + \rho_3.$$

Continuing this process, the solution of $K(y) = 0$ is accomplished by the integration of the normal resolvents

$$\Phi_1 = 0, \ \Phi_2 = 0, \ \ldots, \ \Phi_m = 0,$$

of orders

$$\rho_1, \ \rho_2, \ \ldots, \ \rho_m,$$

respectively. When an integral of the last normal resolvent is adjoined, and the domain becomes

$$\mathcal{D}<x, f, t_1, \ldots, t_m>,$$

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* Vessiot, *loc. cit.*, p. 208.

† Vessiot, *loc. cit.*, p. 223.

the rationality group falls to the identical transformation. The index δ_m becomes

$$\delta_m = \delta + \rho_1 + \rho_2 + \cdots + \rho_m;$$

and, since

$$\delta = n^2 - r,$$

$$\rho_1 + \rho_2 + \cdots + \rho_m = (r - r_1) + (r_1 - r_2) + \cdots + (r_{m-2} - r_{m-1}) + (r_{m-1} - 0) = r,$$

we have

$$\delta_m = n^2,$$

and the differential equation is integrated.

2°. Between each pair of consecutive groups $\Gamma^{(k-1)}$ and $\Gamma^{(k)}$ of the composition series

$$\Gamma^{(0)}, \Gamma^{(1)}, \ldots, \Gamma^{(m-1)}, \Gamma^{(m)},$$

we insert a group $\Lambda^{(k)}$, of σ_k parameters, which is the maximal subgroup of $\Gamma^{(k-1)}$ containing $\Gamma^{(k)}$. Let u_k be a natural transcendental belonging to $\Lambda^{(k)}$. The function u_k, when transformed by $\Gamma^{(k-1)}$, contains $r_{k-1} - \sigma_k$ effective parameters, and therefore satisfies a minimal resolvent

$$\Psi_k(u) = 0, \qquad (k = 1 \ldots m),$$

of order $r_{k-1} - \sigma_k$.

The adjunction of the general integral U_1 of $\Psi_1 = 0$ to the domain $\mathfrak{D} < x, f >$ lowers the rationality group of $K = 0$ from $\Gamma^{(0)}$ to $\Gamma^{(1)}$, the largest subgroup of $\Lambda^{(1)}$ which is self-conjugate in $\Gamma^{(0)}$.* The index of the equation $K = 0$ is therefore increased from δ to $\delta + \rho_1$.

To the domain $\mathfrak{D} < x, f, U_1 >$ we adjoin U_2, the general integral of $\Psi_2 = 0$, and the index of the equation $K = 0$ is therefore increased from $\delta + \rho_1$ to $\delta + \rho_1 + \rho_2$.

By continuing this process m times, it is seen that in the domain $\mathfrak{D} < x, f, U_1, \ldots, U_m >$ the group of the equation becomes the identical transformation; its index rises to

$$\delta + \rho_1 + \rho_2 + \cdots + \rho_m = n^2,$$

and $K(y) = 0$ is integrated.

THE UNIVERSITY OF COLORADO, *March*, 1911.

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* E. Vessiot, *loc. cit.*, p. 238.